EMBEDDING THEOREMS FOR THE DUNKL HARMONIC OSCILLATOR

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Abstract. Embedding results of Sobolev type are proved for the Dunkl harmonic oscillator on the line.

1. Introduction

The Dunkl operator T_σ ($\sigma > -1/2$) on $C^\infty(\mathbb{R})$, is the perturbation of $\frac{d}{dx}$ defined by $T_\sigma = \frac{d}{dx}$ on even functions and $T_\sigma = \frac{d}{dx} + 2\sigma\frac{1}{x}$ on odd functions. This kind of operator, more generally on $\mathbb{R}^n$, was introduced by C.F. Dunkl [12, 13, 14, 15, 16]. It gave rise to what is now called Dunkl theory (see the survey article [32]). In particular, the Dunkl harmonic oscillator [30, 17, 28, 27] is the perturbation $L_\sigma = -T_\sigma^2 + s^2x^2$ ($s > 0$) of the usual harmonic oscillator $H = -\frac{d^2}{dx^2} + s^2x^2$.

On the other hand, let p_k be the sequence of orthogonal polynomials for the measure $e^{-sx^2}|x|^{2\sigma}\,dx$, taken with norm one and positive leading coefficient. Up to normalization, these are the generalized Hermite polynomials [33, p. 380, Problem 25]; see also [9, 11, 18, 10, 30, 31]. The corresponding generalized Hermite functions are $\phi_k = p_k e^{-sx^2/2}$.

For each $m \in \mathbb{N}$, let $\mathcal{S}^m$ be the Banach space of functions $\phi \in C^m(\mathbb{R})$ with $\sup_x |x^i\phi^{(j)}(x)| < \infty$ for $i + j \le m$; thus $\mathcal{S} = \bigcap_m \mathcal{S}^m$, with the corresponding Fréchet topology, is the Schwartz space on $\mathbb{R}$. It is known that L_σ, with domain $\mathcal{S}$, is essentially self-adjoint in $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$, and the spectrum of its self-adjoint extension, denoted by $\mathcal{L}_\sigma$, consists of the eigenvalues $(2k + 1 + 2\sigma)s$ ($k \in \mathbb{N}$), with corresponding eigenfunctions ϕ_k [30]. For each real $m \ge 0$, let W_σ^m be the Hilbert space completion of $\mathcal{S}$ with respect to the scalar product defined by $\langle \phi, \psi\rangle_{W_\sigma^m} = \langle (1+\mathcal{L}_\sigma)^m\phi, \psi\rangle_\sigma$, where $\langle\ ,\ \rangle_\sigma$ denotes the scalar product of $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$. Let also $W_\sigma^\infty = \bigcap_m W_\sigma^m$ with the corresponding Fréchet topology. The subindex ev/odd is added to any space of functions on $\mathbb{R}$ to indicate its subspace of even/odd functions. The following embedding theorems are shown; the second one is of Sobolev type.

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Theorem 1.1. *For each $m \geq 0$, $\mathcal{S}_{\mathrm{ev/odd}}^{M_{m,\mathrm{ev/odd}}} \subset W_{\sigma,\mathrm{ev/odd}}^m$ continuously, where*

$$M_{m,\mathrm{ev/odd}} = \begin{cases} \frac{3\lceil m\rceil+3}{2} + \frac{\lceil m\rceil+1}{4}\lceil\sigma\rceil(\lceil\sigma\rceil+3) + \lceil\sigma\rceil & \text{if } \sigma \geq 0 \text{ and } \lceil m\rceil \text{ is odd} \\ 2\lceil m\rceil + 3 & \text{if } \sigma < 0 \text{ and } \lceil m\rceil \text{ is odd}\,, \end{cases}$$

$$M_{m,\mathrm{ev}} = \begin{cases} \frac{3\lceil m\rceil+2}{2} + \frac{\lceil m\rceil}{4}\lceil\sigma\rceil(\lceil\sigma\rceil+3) + \lceil\sigma\rceil & \text{if } \sigma \geq 0 \text{ and } \lceil m\rceil \text{ is even} \\ 2\lceil m\rceil + 2 & \text{if } \sigma < 0 \text{ and } \lceil m\rceil \text{ is even}\,, \end{cases}$$

$$M_{m,\mathrm{odd}} = \begin{cases} \frac{3\lceil m\rceil+4}{2} + \frac{\lceil m\rceil+2}{4}\lceil\sigma\rceil(\lceil\sigma\rceil+3) + \lceil\sigma\rceil & \text{if } \sigma \geq 0 \text{ and } \lceil m\rceil \text{ is even} \\ 2\lceil m\rceil + 4 & \text{if } \sigma < 0 \text{ and } \lceil m\rceil \text{ is even}\,. \end{cases}$$

Theorem 1.2. *For all $m \in \mathbb{N}$, $W_{\sigma,\mathrm{ev/odd}}^M \subset \mathcal{S}_{\mathrm{ev/odd}}^m$ continuously if $M > M_{m,\mathrm{ev/odd}}$, where*

$$M_{0,\mathrm{ev}} = 2\,, \quad M_{0,\mathrm{odd}} = 5\,, \quad M_{1,\mathrm{ev/odd}} = 6 \qquad (\sigma \leq 0)\,,$$
$$M_{0,\mathrm{ev/odd}} = 6 \qquad (0 < \sigma \leq 1)\,,$$
$$M_{m,\mathrm{ev/odd}} = \begin{cases} m' + 6 & \text{if } m' \text{ is odd/even} \\ m' + 5 & \text{if } m' \text{ is even/odd} \end{cases} \qquad \left(m' = m + \frac{1}{2}\lceil\sigma\rceil(\lceil\sigma\rceil+1) \geq 2\right).$$

Corollary 1.3. *$\mathcal{S} = W_\sigma^\infty$ as Fréchet spaces.*

In other words, Corollary 1.3 states that an element $\phi \in L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$ is in $\mathcal{S}$ if and only if the "Fourier coefficients" $\langle\phi,\phi_k\rangle_\sigma$ are rapidly decreasing on k. This also means that $\mathcal{S} = \bigcap_m \mathcal{D}(\mathcal{L}_\sigma^m)$ ($\mathcal{S}$ is the smooth core of $\mathcal{L}_\sigma$ with the terminology of [6]) because the sequence of eigenvalues of $\mathcal{L}_\sigma$ is in $O(k)$ as $k \to \infty$.

We introduce a version $\mathcal{S}_\sigma^m$ of every $\mathcal{S}^m$, whose definition involves T_σ instead of $\frac{d}{dx}$. They satisfy much simpler embeddings: $S_\sigma^{\lceil m\rceil+1} \subset W_\sigma^m$, and $W_\sigma^{m'} \subset \mathcal{S}_\sigma^m$ if $m' - m > 1$. Even though $\mathcal{S} = \bigcap_m \mathcal{S}_\sigma^m$, the inclusion relations between the spaces $\mathcal{S}_\sigma^m$ and $\mathcal{S}^{m'}$ are complicated, giving rise to the complexity of Theorems 1.1 and 1.2.

Other Sobolev type embedding theorems, for different operators and with different techniques, were recently proved in [36, 37, 38].

Next, we consider other perturbations of H on $\mathbb{R}_+$. Let $\mathcal{S}_{\mathrm{ev},U}$ denote the space of restrictions of even Schwartz functions to some open set U, and set $\phi_{k,U} = \phi_k|_U$.

Theorem 1.4. *Let $P = H - 2f_1\frac{d}{dx} + f_2$, where $f_1 \in C^1(U)$ and $f_2 \in C(U)$ for some open subset $U \subset \mathbb{R}_+$ of full Lebesgue measure. Assume that $f_2 = \sigma(\sigma-1)x^{-2} - f_1^2 - f_1'$ or some $\sigma > -1/2$. Let $h = x^\sigma e^{-F_1}$, where $F_1 \in C^2(U)$ is a primitive of f_1. Then the following properties hold:*

(i) *P, with domain $h\,\mathcal{S}_{\mathrm{ev},U}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, e^{2F_1}\,dx)$;*
(ii) *the spectrum of its self-adjoint extension, denoted by $\mathcal{P}$, consists of the eigenvalues $(4k+1+2\sigma)s$ ($k \in \mathbb{N}$) with multiplicity one and normalized eigenfunctions $\sqrt{2}\,h\phi_{2k,U}$; and*
(iii) *the smooth core of $\mathcal{P}$ is $h\,\mathcal{S}_{\mathrm{ev},U}$.*

This theorem follows by showing that the stated condition on f_1 and f_2 characterizes the cases where P can be obtained by the following process: first, restricting L_σ to even functions, then restricting to U, and finally conjugating by h. The term of P with $\frac{d}{dx}$ can be removed by conjugation with the product of a positive function, obtaining the operator $H + \sigma(\sigma-1)x^{-2}$. In this way, we get all operators of the form $H + cx^{-2}$ with $c > -1/4$.

The conditions of Theorem 1.4 are satisfied by $P = H - 2c_1x^{-1}\frac{d}{dx} + c_2x^{-2}$ ($c_1, c_2 \in \mathbb{R}$) on $\mathbb{R}_+$ if and only if there is some $a \in \mathbb{R}$ such that $a^2+(2c_1-1)a-c_2 = 0$ and $a + c_1 > -1/2$; in this case, $h = x^a$ and $e^{2F_1} = x^{2c_1}$. For some $c_1, c_2 \in \mathbb{R}$, there are two values of a satisfying these conditions, obtaining two different self-adjoint operators defined by P in different Hilbert spaces. For instance, L_σ may define a self-adjoint operator even when $\sigma \leq -1/2$.

This example is applied in [2] to prove a new type of Morse inequalities on strata of compact stratifications [34, 22, 35] with adapted metrics [25, 26, 5], where Witten's perturbation [39] is used for the minimal/maximal ideal boundary conditions of de Rham complex [7, 8, 6]. The version of Morse functions used in [2] is different from those considered by Goresky-MacPherson [20]. More precisely, the operator P describes the radial direction of Witten's perturbed Laplacian in the local conic model of a stratification around each critical point. The two possible choices of a give rise to the minimal/maximal ideal boundary conditions.

Other examples of operators satisfying the conditions of Theorem 1.4 are

$$P = H - 2cx^r\frac{d}{dx} + \sigma(\sigma-1)x^{-2} - c^2x^{2r} - crx^{r-1}\ ,$$
$$P = H - 2c\tan x\,\frac{d}{dx} + \sigma(\sigma-1)x^{-2} + c(c-1)\tan^2 x - c\ ,$$

on $\mathbb{R}_+$ and $\mathbb{R}_+ \setminus (2\mathbb{N}+1)\frac{\pi}{2}$, respectively, where $\sigma > -1/2$, $r \neq -1$ and $c \in \mathbb{R}$.

2. Preliminaries

2.1. **Dunkl operator.** Recall that, for any $\phi \in C^\infty = C^\infty(\mathbb{R})$, there is some $\psi \in C^\infty$ such that $\phi(x) - \phi(0) = x\psi(x)$, which also satisfies

$$\psi^{(m)}(x) = \int_0^1 t^m\phi^{(m+1)}(tx)\,dt \tag{1}$$

for all $m \in \mathbb{N}$ (see e.g. [21, Theorem 1.1.9]). The notation $\psi = x^{-1}\phi$ is used.

The Dunkl operator, in the case of dimension one, is the differential-difference operator T_σ on C^∞, depending on a parameter $\sigma \in \mathbb{R}$, defined by

$$(T_\sigma\phi)(x) = \phi'(x) + 2\sigma\,\frac{\phi(x) - \phi(-x)}{x}\ .$$

It can be considered as a perturbation of the derivative operator $\frac{d}{dx}$.

Consider the decomposition $C^\infty = C^\infty_{\text{ev}} \oplus C^\infty_{\text{odd}}$, as direct sum of subspaces of even and odd functions. The matrix expressions of operators on C^∞ will be considered with respect to this decomposition. The operator of multiplication by a function h will be denoted also by h. We can write $\frac{d}{dx} = \begin{pmatrix} 0 & \frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix}$, $x = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}$ and

$$T_\sigma = \begin{pmatrix} 0 & \frac{d}{dx} + 2\sigma x^{-1} \\ \frac{d}{dx} & 0 \end{pmatrix} = \frac{d}{dx} + 2\sigma\begin{pmatrix} 0 & x^{-1} \\ 0 & 0 \end{pmatrix}$$

on C^∞. With $\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}$, we have

$$[T_\sigma, x] = 1 + 2\Sigma\ , \quad T_\sigma\Sigma + \Sigma T_\sigma = x\,\Sigma + \Sigma\,x = 0\ . \tag{2}$$

Consider the perturbed factorial $m!_\sigma$ of each $m \in \mathbb{N}$, which is inductively defined by setting $0!_\sigma = 1$, and

$$m!_\sigma = \begin{cases} (m-1)!_\sigma m & \text{if } m \text{ is even} \\ (m-1)!_\sigma (m+2\sigma) & \text{if } m \text{ is odd} \end{cases}$$

for $m > 0$. Observe that $m!_\sigma > 0$ if $\sigma > -1/2$. For $k \le m$, even when $k!_\sigma = 0$, the quotient $m!_\sigma / k!_\sigma$ can be understood as the product of the factors from the definition of $m!_\sigma$ which are not included in the definition of $k!_\sigma$. For any $\phi \in C^\infty$ and $m \in \mathbb{N}$, by (1) and induction on m, we get

$$(T_\sigma^m \phi)(0) = \frac{m!_\sigma}{m!} \phi^{(m)}(0) \ . \tag{3}$$

2.2. **Dunkl harmonic oscillator.** Recall that, for dimension one, the harmonic oscillator, and the annihilation and creation operators are $H = -\frac{d^2}{dx^2} + s^2x^2$, $A = sx + \frac{d}{dx}$ and $A' = sx - \frac{d}{dx}$ on C^∞. By using T_σ instead of d/dx, we get perturbations of H, A and A' called Dunkl harmonic oscillator, and Dunkl annihilation and creation operators: $L = -T_\sigma^2 + s^2x^2$, $B = sx + T_\sigma$ and $B' = sx - T_\sigma$. By (2),

$$L = BB' - (1+2\Sigma)s = B'B + (1+2\Sigma)s = \frac{1}{2}(BB' + B'B) \ , \tag{4}$$

$$[L, B] = -2sB \ , \quad [L, B'] = 2sB' \ , \tag{5}$$

$$[B, B'] = 2s(1+2\Sigma) \ , \tag{6}$$

$$[L, \Sigma] = B\Sigma + \Sigma B = B'\Sigma + \Sigma B' = 0 \ . \tag{7}$$

For each[1] $m \in \mathbb{N}$, let $\mathcal{S}^m$ be the space of functions $\phi \in C^\infty$ such that

$$\|\phi\|_{\mathcal{S}^m} = \sum_{i+j \le m} \sup_x |x^i \phi^{(j)}(x)| < \infty \ .$$

This defines a norm $\| \ \|_{\mathcal{S}^m}$ on $\mathcal{S}^m$, which becames a Banach space. We have $\mathcal{S}^{m+1} \subset \mathcal{S}^m$ continuously[2], and $\mathcal{S} = \bigcap_m \mathcal{S}^m$, with the induced Fréchet topology, is the Schwartz space on $\mathbb{R}$. Let us remark that $\|\phi'\|_{\mathcal{S}^m} \le \|\phi\|_{\mathcal{S}^{m+1}}$ for all m.

The above decomposition of C^∞ can be restricted to each $\mathcal{S}^m$ and $\mathcal{S}$, giving $\mathcal{S}^m = \mathcal{S}^m_{\text{ev}} \oplus \mathcal{S}^m_{\text{odd}}$ and $\mathcal{S} = \mathcal{S}_{\text{ev}} \oplus \mathcal{S}_{\text{odd}}$. The matrix expressions of operators on $\mathcal{S}$ will be considered with respect to this decomposition. For $\phi \in C^\infty_{\text{ev}}$, $\psi = x^{-1}\psi$ and $i, j \in \mathbb{N}$, it follows from (1) that

$$|x^i \psi^{(j)}(x)| \le \int_0^1 t^{j-i} |(tx)^i \phi^{(j+1)}(tx)| \, dt \le \sup_{y \in \mathbb{R}} |y^i \phi^{(j+1)}(y)|$$

for all $x \in \mathbb{R}$. Thus $\|\psi\|_{\mathcal{S}^m} \le \|\phi\|_{\mathcal{S}^{m+1}}$ for all $m \in \mathbb{N}$, obtaining that $\mathcal{S}_{\text{odd}} = x\,\mathcal{S}_{\text{ev}}$ and $x^{-1} : C^\infty_{\text{odd}} \to C^\infty_{\text{ev}}$ restricts to a continuous operator $x^{-1} : \mathcal{S}_{\text{odd}} \to \mathcal{S}_{\text{ev}}$. Therefore $x : \mathcal{S}_{\text{ev}} \to \mathcal{S}_{\text{odd}}$ is an isomorphism of Fréchet spaces, and T_σ, B, B' and L define continuous operators on $\mathcal{S}$.

Let $\langle \ , \ \rangle_\sigma$ and $\| \ \|_\sigma$ denote the scalar product and norm of $L^2(\mathbb{R}, |x|^{2\sigma} \, dx)$. Assume from now on that $\sigma > -1/2$, and therefore $\mathcal{S}$ is dense in $L^2(\mathbb{R}, |x|^{2\sigma} \, dx)$. In $L^2(\mathbb{R}, |x|^{2\sigma} \, dx)$, with domain $\mathcal{S}$, $-T_\sigma$ is adjoint of T_σ, B' is adjoint of B, and L is essentially self-adjoint. The self-adjoint extension of L, with domain $\mathcal{S}$, is denoted

[1]We adopt the convention $0 \in \mathbb{N}$.

[2]Let X and Y be topological vector spaces. It is said that $X \subset Y$ continuously if X is a linear subspace of Y and the inclusion map $X \hookrightarrow Y$ is continuous.

by $\mathcal{L}$, or $\mathcal{L}_\sigma$. Its spectrum consists of the eigenvalues $(2k+1+2\sigma)s$ ($k \in \mathbb{N}$). The corresponding normalized eigenfunctions ϕ_k are inductively defined by

$$\phi_0 = s^{(2\sigma+1)/4}\Gamma(\sigma+1/2)^{-1/2}e^{-sx^2/2}\ , \tag{8}$$

$$\phi_k = \begin{cases} (2ks)^{-1/2}B'\phi_{k-1} & \text{if } k \text{ is even} \\ (2(k+2\sigma)s)^{-1/2}B'\phi_{k-1} & \text{if } k \text{ is odd} \end{cases} \tag{9}$$

for $k \geq 1$. We also have

$$B\phi_0 = 0\ , \tag{10}$$

$$B\phi_k = \begin{cases} (2ks)^{1/2}\phi_{k-1} & \text{if } k \text{ is even} \\ (2(k+2\sigma)s)^{1/2}\phi_{k-1} & \text{if } k \text{ is odd} \end{cases} \tag{11}$$

for $k \geq 1$. These assertions follow from (4)–(7) like in the case of H.

2.3. **Generalized Hermite polynomials.** From (8), (9) and the definition of B', it follows that $\phi_k = p_k e^{-sx^2/2}$, where p_k is the sequence of polynomials inductively defined by $p_0 = s^{(2\sigma+1)/4}\Gamma(\sigma+1/2)^{-1/2}$ and

$$p_k = \begin{cases} (2ks)^{-1/2}(2sxp_{k-1} - T_\sigma p_{k-1}) & \text{if } k \text{ is even} \\ (2(k+2\sigma)s)^{-1/2}(2sxp_{k-1} - T_\sigma p_{k-1}) & \text{if } k \text{ is odd} \end{cases} \tag{12}$$

for $k \geq 1$. Up to normalization, these are the generalized Hermite polynomials; i.e., the orthogonal polynomials associated with the measure $|x|^{2\sigma}e^{-sx^2}\,dx$ [33, p. 380, Problem 25], and therefore ϕ_k are the generalized Hermite functions. Each p_k is of precise degree k, even/odd if k is even/odd, and with positive leading coefficient. We also have $T_\sigma p_0 = 0$ and

$$T_\sigma p_k = \begin{cases} (2ks)^{1/2}p_{k-1} & \text{if } k \text{ is even} \\ (2(k+2\sigma)s)^{1/2}p_{k-1} & \text{if } k \text{ is odd} \end{cases} \tag{13}$$

for $k \geq 1$. The following recursion formula follows directly from (12) and (13):

$$p_k = \begin{cases} k^{-1/2}\big((2s)^{1/2}xp_{k-1} - (k-1+2\sigma)^{1/2}p_{k-2}\big) & \text{if } k \text{ is even} \\ (k+2\sigma)^{-1/2}\big((2s)^{1/2}xp_{k-1} - (k-1)^{1/2}p_{k-2}\big) & \text{if } k \text{ is odd}\ . \end{cases} \tag{14}$$

From (14) and by induction on k, we get the following when k is odd[3]

$$x^{-1}p_k = \sum_{\ell\in\{0,2,\dots,k-1\}} (-1)^{\frac{k-\ell-1}{2}} \sqrt{\frac{(k-1)(k-3)\cdots(\ell+2)2s}{(k+2\sigma)(k-2+2\sigma)\cdots(\ell+1+2\sigma)}}\, p_\ell\ . \tag{15}$$

The following theorem contains a simplified version of the asymptotic estimates satisfied by the functions ϕ_k and $\xi_k = |x|^\sigma\phi_k$. They can be obtained by expressing the generalized Hermite polynomials in terms of the Laguerre ones [31, 32], and using the asymptotic estimates of the Laguerre polynomials [19, 3, 24, 23]. The method of Bonan-Clark [4] can be also applied [1].

Theorem 2.1. *There exist $C, C', C'' > 0$, depending on σ and s, such that:*

(i) *if k is odd or $\sigma \geq 0$, then $\xi_k^2(x) \leq C'k^{-1/6}$ for all $x \in \mathbb{R}$;*

(ii) *if k is even and positive, and $\sigma < 0$, then $\xi_k^2(x) \leq C''k^{-1/6}$ for $|x| \geq 1$; and*

[3] As a convention, the product of an empty set of factors is 1. Thus $(k-1)(k-3)\cdots(\ell+2) = 1$ for $\ell = k-1$ in (15).

(iii) *if* $\sigma < 0$, *then* $\phi_k^2(x) \le C''$ *for all* k *and* $|x| \le 1$.

3. Perturbed Schwartz space

We introduce a perturbed version $\mathcal{S}_\sigma^m$ of each $\mathcal{S}^m$ that will be appropriate to show our embedding results. Since $\mathcal{S}_\sigma^m$ must contain the functions ϕ_k, Theorem 2.1 indicates that different definitions must be given for $\sigma \ge 0$ and $\sigma < 0$.

When $\sigma \ge 0$, for any $\phi \in C^\infty$ and $m \in \mathbb{N}$, let

$$\|\phi\|_{\mathcal{S}_\sigma^m} = \sum_{i+j \le m} \sup_x |x|^\sigma \left| x^i T_\sigma^j \phi(x) \right| . \tag{16}$$

This defines a norm $\| \ \|_{\mathcal{S}_\sigma^m}$ on the linear space of functions $\phi \in C^\infty$ with $\|\phi\|_{\mathcal{S}_\sigma^m} < \infty$, and let $\mathcal{S}_\sigma^m$ denote the corresponding Banach space completion. There are direct sum decompositions into subspaces of even and odd functions, $\mathcal{S}_\sigma^m = \mathcal{S}_{\sigma,\mathrm{ev}}^m \oplus \mathcal{S}_{\sigma,\mathrm{odd}}^m$.

When $\sigma < 0$, the even and odd functions are considered separately: let

$$\begin{aligned}\|\phi\|_{\mathcal{S}_\sigma^m} = \sum_{i+j \le m,\ i+j \text{ even}} &\left(\sup_{|x| \le 1} |x^i (T_\sigma^j \phi)(x)| + \sup_{|x| \ge 1} |x|^\sigma \, |x^i (T_\sigma^j \phi)(x)| \right) \\ &+ \sum_{i+j \le m,\ i+j \text{ odd}} \sup_{x \ne 0} |x|^\sigma \, |x^i (T_\sigma^j \phi)(x)| \end{aligned} \tag{17}$$

for $\phi \in C_{\mathrm{ev}}^\infty$, and

$$\begin{aligned}\|\phi\|_{\mathcal{S}_\sigma^m} = \sum_{i+j \le m,\ i+j \text{ even}} &\sup_{x \ne 0} |x|^\sigma \, |x^i (T_\sigma^j \phi)(x)| \\ &+ \sum_{i+j \le m,\ i+j \text{ odd}} \left(\sup_{|x| \le 1} |x^i (T_\sigma^j \phi)(x)| + \sup_{|x| \ge 1} |x|^\sigma \, |x^i (T_\sigma^j \phi)(x)| \right) \end{aligned} \tag{18}$$

for $\phi \in C_{\mathrm{odd}}^\infty$. These expressions define a norm $\| \ \|_{\mathcal{S}_\sigma^m}$ on the linear spaces of functions ϕ in C_{odd}^∞ and C_{ev}^∞ with $\|\phi\|_{\mathcal{S}_\sigma^m} < \infty$. The corresponding Banach space completions will be denoted by $\mathcal{S}_{\sigma,\mathrm{odd}}^m$ and $\mathcal{S}_{\sigma,\mathrm{ev}}^m$. Let $\mathcal{S}_\sigma^m = \mathcal{S}_{\sigma,\mathrm{ev}}^m \oplus \mathcal{S}_{\sigma,\mathrm{odd}}^m$.

Independently of the sign of σ, there are continuous inclusions $\mathcal{S}_\sigma^{m+1} \subset \mathcal{S}_\sigma^m$, and a perturbed Schwartz space is defined as $\mathcal{S}_\sigma = \bigcap_m \mathcal{S}_\sigma^m$, with the corresponding Fréchet topology, which decomposes as direct sum of the subspaces of even and odd functions, $\mathcal{S}_\sigma = \mathcal{S}_{\sigma,\mathrm{ev}} \oplus \mathcal{S}_{\sigma,\mathrm{odd}}$; in particular, $\mathcal{S}_0 = \mathcal{S}$. It easily follows that $\mathcal{S}_\sigma$ consists of functions that are C^∞ on $\mathbb{R} \setminus \{0\}$ but *a priori* possibly not even defined at zero, and $\mathcal{S}_\sigma^m \cap C^\infty$ is dense in $\mathcal{S}_\sigma^m$ for all m; thus $\mathcal{S}_\sigma \cap C^\infty$ is dense in $\mathcal{S}_\sigma$.

Obviously, Σ defines a bounded operator on each $\mathcal{S}_\sigma^m$. It is also easy to see that T_σ defines a bounded operator $\mathcal{S}_\sigma^{m+1} \to \mathcal{S}_\sigma^m$ for any m; notice that, when $\sigma < 0$, the role played by the parity of $i + j$ fits well to prove this property. Similarly, x defines a bounded operator $\mathcal{S}_\sigma^{m+1} \to \mathcal{S}_\sigma^m$ for any m because

$$[T_\sigma^j, x] = \begin{cases} j T_\sigma^{j-1} & \text{if } j \text{ is even} \\ (j + 2\Sigma) T_\sigma^{j-1} & \text{if } j \text{ is odd} \end{cases}$$

by (2). So B and B' define bounded operators $\mathcal{S}_\sigma^{m+1} \to \mathcal{S}_\sigma^m$ too, and L defines a bounded operator $\mathcal{S}_\sigma^{m+2} \to \mathcal{S}_\sigma^m$. Therefore T_σ, x, Σ, B, B' and L define continuous operators on $\mathcal{S}_\sigma$.

In order to prove Theorems 1.1 and 1.2, we introduce an intermediate weakly perturbed Schwartz space $\mathcal{S}_{w,\sigma}$. Like $\mathcal{S}_\sigma$, it is defined as a Fréchet space of the form

$\mathcal{S}_{w,\sigma} = \bigcap_m \mathcal{S}^m_{w,\sigma}$, where each $\mathcal{S}^m_{w,\sigma}$ is the Banach space defined like $\mathcal{S}^m_\sigma$ by using $\frac{d}{dx}$ instead of T_σ in the right hand sides of (16)–(18); in particular, $\mathcal{S}^0_{w,\sigma} = \mathcal{S}^0_\sigma$ as Banach spaces. The notation $\| \ \|_{\mathcal{S}^m_{w,\sigma}}$ will be used for the norm of $\mathcal{S}^m_{w,\sigma}$. As before, $\mathcal{S}_{w,\sigma}$ consists of functions which are C^∞ on $\mathbb{R} \setminus \{0\}$ but *a priori* possibly not even defined at zero, $\mathcal{S}_{w,\sigma} \cap C^\infty$ is dense in $\mathcal{S}_{w,\sigma}$, there is a canonical decomposition $\mathcal{S}_{w,\sigma} = \mathcal{S}_{w,\sigma,\mathrm{ev}} \oplus \mathcal{S}_{w,\sigma,\mathrm{odd}}$, and $\frac{d}{dx}$ and x define bounded operators on $\mathcal{S}^{m+1}_{w,\sigma} \to \mathcal{S}^m_{w,\sigma}$. Thus $\frac{d}{dx}$ and x define continuous operators on $\mathcal{S}_{w,\sigma}$.

Lemma 3.1. *If $\sigma \geq 0$, then $\mathcal{S}^{m+\lceil\sigma\rceil} \subset \mathcal{S}^m_{w,\sigma}$ continuously for all m.*

Proof. Let $\phi \in \mathcal{S}$. For all i and j, we have $|x|^\sigma |x^i \phi^{(j)}(x)| \leq |x^{i+\lceil\sigma\rceil}\phi^{(j)}(x)|$ for $|x| \geq 1$, and $|x|^\sigma |x^i\phi^{(j)}(x)| \leq |x^i\phi^{(j)}(x)|$ for $|x| \leq 1$. So $\|\phi\|_{\mathcal{S}^m_{w,\sigma}} \leq \|\phi\|_{\mathcal{S}^{m+\lceil\sigma\rceil}}$. □

Lemma 3.2. *If $\sigma \geq 0$, then $\mathcal{S}^{m'}_{w,\sigma} \subset \mathcal{S}^m$ continuously for all m, where*

$$m' = m + 1 + \frac{1}{2}\lceil\sigma\rceil(\lceil\sigma\rceil + 1)$$

Proof. Let $\phi \in \mathcal{S}_{w,\sigma}$. For all i and j,

$$\left|x^i\phi^{(j)}(x)\right| \leq |x|^\sigma \left|x^i\phi^{(j)}(x)\right| \tag{19}$$

for $|x| \geq 1$. It remains to prove an inequality of this type for $|x| \leq 1$, which will be a consequence of the following assertion.

Claim 1. For each $n \in \mathbb{N}$, there are finite families of real numbers, $c^n_{a,b}$, $d^n_{k,\ell}$ and $e^n_{u,v}$, where the indices a, b, k, ℓ, u and v run in finite subsets of $\mathbb{N}$ with $b, \ell, v \leq M_n = 1 + \frac{n(n+1)}{2}$ and $k \geq n$, such that, for all $\phi \in C^\infty$,

$$\phi(x) = \sum_{a,b} c^n_{a,b} x^a \phi^{(b)}(1) + \sum_{k,\ell} d^n_{k,\ell} x^k \phi^{(\ell)}(x) + \sum_{u,v} e^n_{u,v} x^u \int_x^1 t^n \phi^{(v)}(t)\, dt\ .$$

Assuming that Claim 1 is true, the proof can be completed as follows. Let $\phi \in \mathcal{S}_{w,\sigma}$ and set $n = \lceil\sigma\rceil$. For $|x| \leq 1$, according to Claim 1, $|\phi(x)|$ is bounded by

$$\begin{aligned}
&\sum_{a,b} |c^n_{a,b}| \left|\phi^{(b)}(1)\right| + \sum_{k,\ell} |d^n_{k,\ell}| \left|x^k\phi^{(\ell)}(x)\right| + \sum_{u,v} |e^n_{u,v}|\, 2 \max_{|t|\leq 1} \left|t^n \phi^{(v)}(t)\right| \\
&\quad \leq \sum_{i,j} |c^n_{a,b}| \left|\phi^{(b)}(1)\right| + \sum_{k,\ell} |d^n_{k,\ell}|\, |x|^\sigma \left|\phi^{(\ell)}(x)\right| + \sum_{u,v} |e^n_{u,v}|\, 2 \max_{|t|\leq 1} |t|^\sigma \left|\phi^{(v)}(t)\right|\ .
\end{aligned}$$

Let $m, i, j \in \mathbb{N}$ with $i + j \leq m$. By applying the above inequality to the function $x^i\phi^{(j)}$, and expressing each derivative $(x^i\phi^{(j)})^{(r)}$ as a linear combination of functions of the form $x^p\phi^{(q)}$ with $p+q \leq i+j+r$, it follows that there is some $C \geq 1$, depending only on σ and m, such that

$$\left|x^i\phi^{(j)}(x)\right| \leq C\, \|\phi\|_{\mathcal{S}^{i+j+M_n}_{w,\sigma}} \tag{20}$$

for $|x| \leq 1$. By (19) and (20), $\|\phi\|_{\mathcal{S}^m} \leq C\, \|\phi\|_{\mathcal{S}^{m'}_{w,\sigma}}$ with $m' = m + M_n$.

Now, let us prove Claim 1. By induction on n and using integration by parts, it is easy to prove that

$$\int_x^1 t^n \phi^{(n+1)}(t)\, dt = \sum_{r=0}^n (-1)^{n-r}\, \frac{n!}{r!} \left(\phi^{(r)}(1) - x^r\phi^{(r)}(x)\right)\ . \tag{21}$$

This shows directly Claim 1 for $n \in \{0,1\}$. Proceeding by induction, let $n > 1$ and assume that Claim 1 holds for $n-1$. By (21), it is enough to find appropriate expressions of $x^r\phi^{(r)}(x)$ for $0 < r < n$. For that purpose, apply Claim 1 for $n-1$ to each function $\phi^{(r)}$, and multiply the resulting equality by x^r to get

$$x^r\phi^{(r)}(x) = \sum_{a,b} c^{n-1}_{a,b} x^{r+a}\phi^{(r+b)}(1) + \sum_{k,\ell} d^{n-1}_{k,\ell} x^{r+k}\phi^{(r+\ell)}(x) + \sum_{u,v} e^{n-1}_{u,v} x^{r+u} \int_x^1 t^{n-1}\phi^{(r+v)}(t)\,dt\ ,$$

where a, b, k, ℓ, u and v run in finite subsets of $\mathbb{N}$ with $b, \ell, v \leq M_{n-1}$ and $k \geq n-1$; thus $r+k \geq n$ and

$$r+b, r+\ell, r+v \leq n-1+M_{n-1} = M_n - 1\ .$$

Therefore it only remains to rise the exponent of t by a unit in the integrals of the last sum. Once more, integration by parts makes the job:

$$\int_x^1 t^n\phi^{(r+v+1)}(t)\,dt = \phi^{(r+v)}(1) - x^n\phi^{(r+v)}(x) - n\int_x^1 t^{n-1}\phi^{(r+v)}\,dt\ . \quad \square$$

Lemma 3.3. *If $\sigma < 0$, then $\mathcal{S}^{m+1} \subset \mathcal{S}^m_{w,\sigma}$ continuously for all m.*

Proof. This is proved by induction on m. For $\phi \in C^\infty_{\mathrm{ev}}$ and $|x| \geq 1$, we have $|x|^\sigma\,|\phi(x)| \leq |\phi(x)|$, obtaining $\|\ \|_{\mathcal{S}^0_{w,\sigma}} \leq \|\ \|_{\mathcal{S}^0}$ on C^∞_{ev}. On the other hand, for $\phi \in C^\infty_{\mathrm{odd}}$ and $\psi = x^{-1}\phi \in C^\infty_{\mathrm{ev}}$, we get

$$|x|^\sigma|\phi(x)| \leq \begin{cases} |\psi(x)| & \text{if } 0 < |x| \leq 1 \\ |\phi(x)| & \text{if } |x| \geq 1\ . \end{cases}$$

So, by (1),

$$\|\phi\|_{\mathcal{S}^0_{w,\sigma}} \leq \max\{\|\phi\|_{\mathcal{S}^0}, \|\psi\|_{\mathcal{S}^0}\} \leq \|\phi\|_{\mathcal{S}^1}\ .$$

Now, assume that $m > 0$ and the result holds for $m-1$. Let $i, j \in \mathbb{N}$ such that $i+j \leq m$, and let $\phi \in \mathcal{S}_{\mathrm{ev}} \cup \mathcal{S}_{\mathrm{odd}}$. Independently of the parity of ϕ and $i+j$, we have $|x|^\sigma\,|x^i\phi^{(j)}(x)| \leq |x^i\phi^{(j)}(x)|$ for $|x| \geq 1$.

Suppose that $\phi \in \mathcal{S}_{\mathrm{ev}}$. If $i = 0$ and j is odd, then $\phi^{(j)} \in \mathcal{S}_{\mathrm{odd}}$. Thus there is some $\psi \in \mathcal{S}_{\mathrm{ev}}$ such that $\phi^{(j)} = x\psi$, obtaining $|x|^\sigma|\phi^{(j)}(x)| \leq |\psi(x)|$ for $0 < |x| \leq 1$. If $i+j$ is odd and $i > 0$, then $|x|^\sigma|x^i\phi^{(j)}(x)| \leq |x^{i-1}\phi^{(j)}(x)|$ for $0 < |x| \leq 1$. Hence, by (1), there is some $C > 0$, independent of ϕ, such that

$$\|\phi\|_{\mathcal{S}^m_{w,\sigma}} \leq C\max\{\|\phi\|_{\mathcal{S}^m}, \|\psi\|_{\mathcal{S}^0}\} \leq C\max\bigl\{\|\phi\|_{\mathcal{S}^m}, \|\phi^{(j)}\|_{\mathcal{S}^1}\bigr\} \leq C\,\|\phi\|_{\mathcal{S}^{m+1}}\ .$$

Finally, assume $\phi \in \mathcal{S}_{\mathrm{odd}}$. There is some $\psi \in \mathcal{S}_{\mathrm{ev}}$ such that $\phi = x\psi$. If i is even and $j = 0$, then $|x|^\sigma\,|x^i\phi(x)| \leq |x^i\psi(x)|$ for $0 < |x| \leq 1$. If $i+j$ is even and $j > 0$, then

$$|x|^\sigma\left|x^i\phi^{(j)}(x)\right| \leq \left|x^i\psi^{(j)}(x)\right| + j\,|x|^\sigma\left|x^i\psi^{(j-1)}(x)\right|$$

for $0 < |x| \leq 1$ because $[\frac{d^j}{dx^j}, x] = j\,\frac{d^{j-1}}{dx^{j-1}}$. Therefore, by (1) and the induction hypothesis, there are some $C', C'' > 0$, independent of ϕ, such that

$$\|\phi\|_{\mathcal{S}^m_{w,\sigma}} \leq C'\max\bigl\{\|\phi\|_{\mathcal{S}^m}, \|\psi\|_{\mathcal{S}^m} + \|\psi\|_{\mathcal{S}^{m-1}_{w,\sigma}}\bigr\} \leq C''\,\|\phi\|_{\mathcal{S}^{m+1}}\ . \quad \square$$

Lemma 3.4. *If $\sigma < 0$, then $\mathcal{S}^{m+1}_{w,\sigma} \subset \mathcal{S}^m$ continuously for all m.*

Proof. Let $i,j\in\mathbb{N}$ such that $i+j\le m$. Since

$$\left|x^i\phi^{(j)}(x)\right|\le\begin{cases}|x|^\sigma\left|x^i\phi^{(j)}(x)\right| & \text{if } 0<|x|\le 1\\ |x|^\sigma\left|x^{i+1}\phi^{(j)}(x)\right| & \text{if } |x|\ge 1\ .\end{cases}$$

for any $\phi\in C^\infty$, we get $\|\phi\|_{\mathcal{S}^m}\le\|\phi\|_{\mathcal{S}^{m+1}_{w,\sigma}}$. □

Corollary 3.5. $\mathcal{S}=\mathcal{S}_{w,\sigma}$ *as Fréchet spaces.*

Corollary 3.6. x^{-1} *defines a bounded operator* $\mathcal{S}^{m'}_{w,\sigma,\mathrm{odd}}\to\mathcal{S}^m_{w,\sigma,\mathrm{ev}}$*, where*

$$m'=\begin{cases}m+2+\frac{1}{2}\lceil\sigma\rceil(\lceil\sigma\rceil+3) & \text{if } \sigma\ge 0\\ m+3 & \text{if } \sigma<0\ .\end{cases}$$

Proof. If $\sigma\ge 0$, the composite

$$\mathcal{S}^{m+2+\frac{1}{2}\lceil\sigma\rceil(\lceil\sigma\rceil+3)}_{w,\sigma,\mathrm{odd}}\hookrightarrow\mathcal{S}^{m+\lceil\sigma\rceil+1}_{\mathrm{odd}}\xrightarrow{x^{-1}}\mathcal{S}^{m+\lceil\sigma\rceil}_{\mathrm{ev}}\hookrightarrow\mathcal{S}^m_{w,\sigma,\mathrm{ev}}$$

is bounded by Lemmas 3.1 and 3.2. If $\sigma<0$, the composite

$$\mathcal{S}^{m+3}_{w,\sigma,\mathrm{odd}}\hookrightarrow\mathcal{S}^{m+2}_{\mathrm{odd}}\xrightarrow{x^{-1}}\mathcal{S}^{m+1}_{\mathrm{ev}}\hookrightarrow\mathcal{S}^m_{w,\sigma,\mathrm{ev}}\ ,$$

is bounded by Lemmas 3.3 and 3.4. □

Corollary 3.7. x^{-1} *defines a continuous operator* $\mathcal{S}_{w,\sigma,\mathrm{odd}}\to\mathcal{S}_{w,\sigma,\mathrm{ev}}$.

Lemma 3.8. $\mathcal{S}^{M_{m,\mathrm{ev/odd}}}_{w,\sigma,\mathrm{ev/odd}}\subset\mathcal{S}^m_{\sigma,\mathrm{ev/odd}}$ *continuously for all* m*, where*

$$M_{m,\mathrm{ev/odd}}=\begin{cases}\frac{3m}{2}+\frac{m}{4}\lceil\sigma\rceil(\lceil\sigma\rceil+3) & \text{if } \sigma\ge 0 \text{ and } m \text{ is even}\\ 2m & \text{if } \sigma<0 \text{ and } m \text{ is even}\ ,\end{cases}$$

$$M_{m,\mathrm{ev}}=\begin{cases}\frac{3m-1}{2}+\frac{m-1}{4}\lceil\sigma\rceil(\lceil\sigma\rceil+3) & \text{if } \sigma\ge 0 \text{ and } m \text{ is odd}\\ 2m-1 & \text{if } \sigma<0 \text{ and } m \text{ is odd}\ ,\end{cases}$$

$$M_{m,\mathrm{odd}}=\begin{cases}\frac{3m+1}{2}+\frac{m+1}{4}\lceil\sigma\rceil(\lceil\sigma\rceil+3) & \text{if } \sigma\ge 0 \text{ and } m \text{ is odd}\\ 2m+1 & \text{if } \sigma<0 \text{ and } m \text{ is odd}\ .\end{cases}$$

Proof. The result follows by induction on m. The statement is true for $m=0$ because $\mathcal{S}^0_{w,\sigma}=\mathcal{S}^0_\sigma$ as Banach spaces. Now, take any $m>0$, and assume that the result holds for $m-1$.

For $\phi\in C^\infty_{\mathrm{ev}}$, $i+j\le m$ with $j>0$, and $x\in\mathbb{R}$, we have $\left|x^i(T^j_\sigma\phi)(x)\right|=\left|x^i(T^{j-1}_\sigma\phi')(x)\right|$ with $\phi'\in C^\infty_{\mathrm{odd}}$, obtaining $\|\phi\|_{\mathcal{S}^m_\sigma}\le\|\phi'\|_{\mathcal{S}^{m-1}_\sigma}$. But, by the induction hypothesis and since $M_{m,\mathrm{ev}}=M_{m-1,\mathrm{odd}}+1$, there are some $C,C'>0$, independent of ϕ, such that

$$\|\phi'\|_{\mathcal{S}^{m-1}_\sigma}\le C\,\|\phi'\|_{\mathcal{S}^{M_{m-1,\mathrm{odd}}}_{w,\sigma}}\le C'\,\|\phi\|_{\mathcal{S}^{M_{m,\mathrm{ev}}}_{w,\sigma}}\ .$$

For $\phi\in C^\infty_{\mathrm{odd}}$, let $\psi=x^{-1}\phi$, and take i, j and x as above. Then

$$\left|x^i(T^j_\sigma\phi)(x)\right|\le\left|x^i(T^{j-1}_\sigma\phi')(x)\right|+2\,|\sigma|\left|x^i(T^{j-1}_\sigma\psi)(x)\right|$$

with $\phi',\psi\in C^\infty_{\mathrm{ev}}$, obtaining $\|\phi\|_{\mathcal{S}^m_\sigma}\le\|\phi'\|_{\mathcal{S}^{m-1}_\sigma}+2\,|\sigma|\,\|\psi\|_{\mathcal{S}^{m-1}_\sigma}$. But, by the induction hypothesis, Corollary 3.6, and since

$$M_{m,\mathrm{odd}}=\begin{cases}M_{m-1,\mathrm{ev}}+2+\frac{1}{2}\lceil\sigma\rceil(\lceil\sigma\rceil+3) & \text{if } \sigma\ge 0\\ M_{m-1,\mathrm{ev}}+3 & \text{if } \sigma<0\ ,\end{cases}$$

there are some $C, C' > 0$, independent of ϕ, such that

$$\begin{aligned}\|\phi'\|_{\mathcal{S}_\sigma^{m-1}} + 2\,|\sigma|\,\|\psi\|_{\mathcal{S}_\sigma^{m-1}} &\le C\left(\|\phi'\|_{\mathcal{S}_{w,\sigma}^{M_{m-1,\mathrm{ev}}}} + \|\psi\|_{\mathcal{S}_{w,\sigma}^{M_{m-1,\mathrm{ev}}}}\right)\\ &\le C'\,\|\phi\|_{\mathcal{S}_{w,\sigma}^{M_{m,\mathrm{odd}}}}\,. \quad \square\end{aligned}$$

Corollary 3.9. *$\mathcal{S}_{w,\sigma} \subset \mathcal{S}_\sigma$ continuously.*

Corollary 3.10. *$\mathcal{S}_{\mathrm{ev/odd}}^{M_{m,\mathrm{ev/odd}}} \subset \mathcal{S}_{\sigma,\mathrm{ev/odd}}^m$ continuously for all m, where*

$$M_{m,\mathrm{ev/odd}} = \begin{cases} \frac{3m}{2} + \frac{m}{4}\lceil\sigma\rceil(\lceil\sigma\rceil+3) + \lceil\sigma\rceil & \text{if } \sigma \ge 0 \text{ and } m \text{ is even}\\ 2m+1 & \text{if } \sigma < 0 \text{ and } m \text{ is even}\,,\end{cases}$$

$$M_{m,\mathrm{ev}} = \begin{cases} \frac{3m-1}{2} + \frac{m-1}{4}\lceil\sigma\rceil(\lceil\sigma\rceil+3) + \lceil\sigma\rceil & \text{if } \sigma \ge 0 \text{ and } m \text{ is odd}\\ 2m & \text{if } \sigma < 0 \text{ and } m \text{ is odd}\,,\end{cases}$$

$$M_{m,\mathrm{odd}} = \begin{cases} \frac{3m+1}{2} + \frac{m+1}{4}\lceil\sigma\rceil(\lceil\sigma\rceil+3) + \lceil\sigma\rceil & \text{if } \sigma \ge 0 \text{ and } m \text{ is odd}\\ 2m+2 & \text{if } \sigma < 0 \text{ and } m \text{ is odd}\,.\end{cases}$$

Proof. This follows from Lemmas 3.1, 3.3 and 3.8. $\square$

4. Perturbed Sobolev spaces

Observe that $\mathcal{S}_\sigma \subset L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$. Like in the case where $\mathcal{S}$ is considered as domain, it is easy to check that, in $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$, with domain $\mathcal{S}_\sigma$, B is adjoint of B' and L is symmetric.

Lemma 4.1. *$\mathcal{S}_\sigma$ is a core[4] of $\mathcal{L}$.*

Proof. Let R denote the restriction of $\mathcal{L}$ to $\mathcal{S}_\sigma$. Then $\mathcal{L} \subset \overline{R} \subset R^* \subset \mathcal{L}^* = \mathcal{L}$ in $L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$ because $\mathcal{S} \subset \mathcal{S}_\sigma$ by Corollaries 3.5 and 3.9. $\square$

For each $m \ge 0$, let W_σ^m be the Hilbert space completion of $\mathcal{S}$ with respect to the scalar product $\langle\ ,\ \rangle_{W_\sigma^m}$ defined by $\langle\phi,\psi\rangle_{W_\sigma^m} = \langle(1+\mathcal{L})^m\phi,\psi\rangle_\sigma$. The corresponding norm will be denoted by $\|\ \|_{W_\sigma^m}$, whose equivalence class is independent of the parameter s used to define L. In particular, $W_\sigma^0 = L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$. As usual, $W_\sigma^{m'} \subset W_\sigma^m$ when $m' > m$, and let $W_\sigma^\infty = \bigcap_m W_\sigma^m$ with the induced Fréchet topology. Once more, there are direct sum decompositions into subspaces of even and odd (generalized) functions, $W_\sigma^m = W_{\sigma,\mathrm{ev}}^m \oplus W_{\sigma,\mathrm{odd}}^m$ ($m \in [0,\infty]$).

Remark 1. By Lemma 4.1, $\mathcal{S}_\sigma$ can be used instead of $\mathcal{S}$ in the definition of W_σ^m.

Obviously, L defines a bounded operator $W_\sigma^{m+2} \to W_\sigma^m$ for each $m \ge 0$, and therefore a continuous operator on W_σ^∞. Moreover, by (7), Σ defines a bounded operator on each W_σ^m, and therefore a continuous operator on W_σ^∞.

Lemma 4.2. *B and B' define bounded operators $W_\sigma^{m+1} \to W_\sigma^m$ for each m.*

Proof. This follows by induction on m. For $m = 0$, by (4),

$$\|B\phi\|_\sigma^2 = \|B'\phi\|_\sigma^2 = \langle B'B\phi,\phi\rangle_\sigma = \langle(L-(1+2\Sigma)s)\phi,\phi\rangle_\sigma \le C_0\,\|\phi\|_{W_\sigma^1}^2$$

for all $\phi \in \mathcal{S}$ and some $C_0 > 0$ independent of ϕ. It follows that B and B' define bounded operators $W_\sigma^1 \to L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$.

[4] Recall that a *core* of a closed densely defined operator T between Hilbert spaces is any subspace of its domain $\mathcal{D}(T)$ which is dense with the graph norm.

Now let $m>0$ and assume that there are $C_{m-1},C'_{m-1}>0$ so that $\|B\phi\|^2_{W^{m-1}_\sigma}\le C_{m-1}\|\phi\|^2_{W^m_\sigma}$ and $\|B'\phi\|^2_{W^{m-1}_\sigma}\le C'_{m-1}\|\phi\|^2_{W^m_\sigma}$ for all $\phi\in\mathcal{S}$. Then, by (5),

$$\begin{aligned}\|B\phi\|^2_{W^m_\sigma}=\langle(1+L)B\phi,B\phi\rangle_{W^{m-1}_\sigma}&=\|B\phi\|^2_{W^{m-1}_\sigma}+\langle LB\phi,B\phi\rangle_{W^{m-1}_\sigma}\\&=(1-2s)\|B\phi\|^2_{W^{m-1}_\sigma}+\langle BL\phi,B\phi\rangle_{W^{m-1}_\sigma}\\&\le(1-2s)\|B\phi\|^2_{W^{m-1}_\sigma}+\|BL\phi\|_{W^{m-1}_\sigma}\|B\phi\|_{W^{m-1}_\sigma}\\&\le(1-2s)C_{m-1}\|\phi\|^2_{W^m_\sigma}+C^2_{m-1}\|L\phi\|_{W^m_\sigma}\|\phi\|_{W^m_\sigma}\le C_m\|\phi\|^2_{W^{m+1}_\sigma}\end{aligned}$$

for some $C_m>0$ independent of ϕ. Similarly, $\|B'\phi\|^2_{W^m_\sigma}\le C'_m\|\phi\|^2_{W^{m+1}_\sigma}$ for some $C'_m>0$ independent of ϕ. □

Σ preserves $W^m_{\sigma,\mathrm{ev/odd}}$, whilst B and B' define maps $W^{m+1}_{\sigma,\mathrm{ev/odd}}\to W^m_{\sigma,\mathrm{odd/ev}}$. Therefore L defines maps $W^{m+2}_{\sigma,\mathrm{ev/odd}}\to W^m_{\sigma,\mathrm{ev/odd}}$ by (4). Observe that B' is not adjoint of B in W^m_σ for $m>0$.

The motivation of our tour through perturbed Schwartz spaces is the following embedding results; the second one is a version of the Sobolev embedding theorem.

Proposition 4.3. *$\mathcal{S}^{\lceil m\rceil+1}_\sigma\subset W^m_\sigma$ continuously.*

Proposition 4.4. *$W^{m'}_\sigma\subset\mathcal{S}^m_\sigma$ continuously if $m'-m>1$.*

Corollary 4.5. *$\mathcal{S}_\sigma=W^\infty_\sigma$ as Fréchet spaces.*

For each non-commutative polynomial p (of two variables, X and Y), let p' denote the non-commutative polynomial obtained by reversing the order of the variables in p; e.g., if $p(X,Y)=X^2Y^3X$, then $p'(X,Y)=XY^3X^2$. It will be said that p is *symmetric* if $p(X,Y)=p'(Y,X)$. Notice that any non-commutative polynomial of the form $p'(Y,X)p(X,Y)$ is symmetric. Given any non-commutative polynomial p, the continuous operators $p(B,B')$ and $p'(B',B)$ on $\mathcal{S}_\sigma$ are adjoint of each other in $L^2(\mathbb{R},|x|^{2\sigma}\,dx)$; thus $p(B,B')$ is a symmetric operator if p is symmetric. The following lemma will be used in the proof of Proposition 4.3

Lemma 4.6. *For each $m\in\mathbb{N}$, we have $(1+L)^m=\sum_a q'_a(B',B)q_a(B,B')$ for some finite family of homogeneous non-commutative polynomials q_a of degree $\le m$.*

Proof. The result follows easily from the following assertions.

Claim 2. If m is even, then $L^m=g_m(B,B')^2$ for some symmetric homogeneous non-commutative polynomial g_m of degree m.

Claim 3. If m is odd, then

$$L^m=g'_{m,1}(B',B)g_{m,1}(B,B')+g'_{m,2}(B',B)g_{m,2}(B,B')$$

for some homogeneous non-commutative polynomials $g_{m,1}$ and $g_{m,2}$ of degree m.

If m is even, then $L^{m/2}=g_m(B,B')$ for some symmetric homogeneous non-commutative polynomial g_m of degree m by (4). So $L^m=g_m(B,B')^2$, showing Claim 2.

If m is odd, write $L^{\lfloor m/2\rfloor}=f_m(B,B')$ as above for some symmetric homogeneous non-commutative polynomial f_m of degree $m-1$. Then, by (4),

$$L^m=\frac{1}{2}f_m(B,B')(BB'+B'B)f_m(B,B')\,.$$

Thus Claim 3 follows with

$$g_{m,1}(B,B')=\frac{1}{\sqrt{2}}\,B'f_m(B,B')\;,\quad g_{m,2}(B,B')=\frac{1}{\sqrt{2}}\,Bf_m(B,B')\;.\quad\square$$

Proof of Proposition 4.3 when $\sigma\ge 0$. By the definitions of B and B', for each non-commutative polynomial p of three variables with degree $\le m'$, there exists some $C_p>0$ such that $|x|^\sigma\,|p(x,B,B')\phi|\le C_p\,\|\phi\|_{\mathcal{S}_\sigma^{m'}}$ for all $\phi\in\mathcal{S}_\sigma$. We can assume that $m\in\mathbb{N}$, and write $(1+L)^m=\sum_a q'_a(B',B)q_a(B,B')$ according to Lemma 4.6. Let $\bar q_a(x,B,B')=x^{m'-m}q_a(B,B')$. Then, for each $\phi\in\mathcal{S}_\sigma$,

$$\begin{aligned}\|\phi\|^2_{W^m_\sigma}=\sum_a\|q_a(B,B')\phi\|^2_\sigma&=\sum_a\int_{-\infty}^{\infty}|(q_a(B,B')\phi)(x)|^2\,|x|^{2\sigma}\,dx\\&\le 2\sum_a\left(C^2_{q_a}+C^2_{\bar q_a}\int_1^\infty x^{-2(m'-m)}\,dx\right)\|\phi\|^2_{\mathcal{S}_\sigma^{m'}}\;,\end{aligned}$$

where the integral is finite because $m'-m\ge 1$. $\square$

Proof of Proposition 4.3 when $\sigma<0$. Now, according to (17) and (18), for each homogeneous non-commutative polynomial p of three variables with degree $d\le m'$, there is some $C_p>0$ such that $|(p(x,B,B')\phi)(x)|\le C_p\,\|\phi\|_{\mathcal{S}_\sigma^{m'}}$ if $|x|\le 1$, $\phi\in\mathcal{S}_{\sigma,\mathrm{ev/odd}}$ and d is even/odd; and $|x|^\sigma\,|(p(x,B,B')\phi)(x)|\le C_p\,\|\phi\|_{\mathcal{S}_\sigma^{m'}}$ otherwise.

With the notation of Lemma 4.6, assuming that $m\in\mathbb{N}$, let d_a denote the degree of each homogenous non-commutative polynomial q_a, and let $\bar q_a(x,B,B')$ be defined like in the case $\sigma\ge 0$. Then, as above, $\|\phi\|^2_{W^m_\sigma}$ is bounded by

$$2\left(\sum_{d_a\text{ even}}C^2_{q_a}\int_0^1 x^{2\sigma}\,dx+\sum_{d_a\text{ odd}}C^2_{q_a}+\sum_a C^2_{\bar q_a}\int_1^\infty x^{-2(m'-m)}\,dx\right)\|\phi\|^2_{\mathcal{S}_\sigma^{m'}}$$

for $\phi\in\mathcal{S}_{\sigma,\mathrm{ev}}$, and by

$$2\left(\sum_{d_a\text{ even}}C^2_{q_a}+\sum_{d_a\text{ odd}}C^2_{q_a}\int_0^1 x^{2\sigma}\,dx+\sum_a C^2_{\bar q_a}\int_1^\infty x^{-2(m'-m)}\,dx\right)\|\phi\|^2_{\mathcal{S}_\sigma^{m'}}$$

for $\phi\in\mathcal{S}_{\sigma,\mathrm{odd}}$, where the integrals are finite because $\sigma>-1/2$ and $m'-m\ge 1$. $\square$

Remark 2. In the last proofs, the condition $m'-m>1/2$ is enough to get finite integrals. But the definition of q_a requires $m\in\mathbb{N}$, and $\bar q_a$ is a polynomial because $m'-m\in\mathbb{N}$.

For $c=(c_k)$ and $c'=(c'_k)$ in $\mathbb{R}^\mathbb{N}$, the expressions $\|c\|_{\mathcal{C}_m}=\sup_k|c_k|(1+k)^m$ and $\langle c,c'\rangle_{\ell^2_m}=\sum_k c_kc'_k(1+k)^m$ define a norm and a scalar product with possible infinite values, and let $\|\ \|_{\ell^2_m}$ denote the norm with possible infinite values induced by $\langle\ ,\ \rangle_{\ell^2_m}$. Then $\mathcal{C}_m=\{\,c\in\mathbb{R}^\mathbb{N}\mid\|c\|_{\mathcal{C}_m}<\infty\,\}$ becomes a Banach space with $\|\ \|_{\mathcal{C}_m}$, and $\ell^2_m=\{\,c\in\mathbb{R}^\mathbb{N}\mid\|c\|_{\ell^2_m}<\infty\,\}$ is a Hilbert space with $\langle\ ,\ \rangle_{\ell^2_m}$. There are continuous inclusions, $\mathcal{C}_{m'}\subset\mathcal{C}_m$ and $\ell^2_{m'}\subset\ell^2_m$ for $m'>m$, and let $\mathcal{C}_\infty=\bigcap_m\mathcal{C}_m$ and $\ell^2_\infty=\bigcap_m\ell^2_m$, with the induced Fréchet topologies. A sequence $c=(c_k)$ will be called even/odd if $c_k=0$ for all odd/even k. There are direct sum decompositions into subspaces of even and odd sequences, $\mathcal{C}_m=\mathcal{C}_{m,\mathrm{ev}}\oplus\mathcal{C}_{m,\mathrm{odd}}$ and $\ell^2_m=\ell^2_{m,\mathrm{ev}}\oplus\ell^2_{m,\mathrm{odd}}$, for $m\in[0,\infty]$.

Lemma 4.7. *$\ell^2_{2m}\subset\mathcal{C}_m$ and $\mathcal{C}_{m'}\subset\ell^2_m$ continuously for all m if $2m'-m>1$.*

Proof. It is easy to see that $\|c\|_{\mathcal{C}_m} \le \|c\|_{\ell^2_{2m}}$ and $\|c\|_{\ell^2_m} \le \|c\|_{\mathcal{C}_{m'}} (\sum_k (1+k)^{m-2m'})^{1/2}$ for any $c \in \mathcal{C}_\infty$, where the last series is convergent because $m - 2m' < -1$. □

Corollary 4.8. *$\ell^2_\infty = \mathcal{C}_\infty$ as Fréchet spaces.*

According to Section 2.2, the "Fourier coefficients" mapping $\phi \mapsto (\langle \phi_k, \phi \rangle_\sigma)$ defines a quasi-isometry $W^m_\sigma \to \ell^2_m$ for all finite m, and therefore an isomorphism $W^\infty_\sigma \to \mathcal{C}_\infty$ of Fréchet espaces. This map is compatible with the decompositions into even and odd subspaces.

Corollary 4.9. *Any $\phi \in L^2(\mathbb{R}, |x|^{2\sigma}\, dx)$ is in $\mathcal{S}_\sigma$ if and only if its "Fourier coefficients" $\langle \phi_k, \phi \rangle_\sigma$ are rapidly degreasing on k.*

Proof. By Corollary 4.5, the "Fourier coefficients" mapping defines an isomorphism $\mathcal{S}_\sigma \to \mathcal{C}_\infty$ of Fréchet spaces. □

Proposition 4.10. *The operator $W^{m'}_\sigma \hookrightarrow W^m_\sigma$ is compact for $m' > m$.*

By using the "Fourier coefficients" mapping, Proposition 4.10 (a version of the Rellich theorem) follows from the following lemma (see e.g. [29, Theorem 5.8]).

Lemma 4.11. *The operator $\ell^2_{m'} \hookrightarrow \ell^2_m$ is compact for $m' > m$.*

Proof of Proposition 4.4. For $\phi \in \mathcal{S}_\sigma$, its "Fourier coefficients" $c_k = \langle \phi_k, \phi \rangle_\sigma$ form a sequence $c = (c_k)$ in $\mathcal{C}_\infty$, and $\sum_k |c_k|\,(1+k)^{m/2} \le \|c\|_{\ell^2_{m'}} (\sum_k (1+k)^{m-m'})^{1/2}$ by Cauchy-Schwartz inequality, where the last series is convergent since $m - m' < -1$. Therefore

$$\sum_k |c_k|\,(1+k)^{m/2} \le C\,\|\phi\|_{W^{m'}_\sigma} \tag{22}$$

for some $C > 0$ independent of ϕ.

On the other hand, for all $i, j \in \mathbb{N}$ with $i + j \le m$, there is some homogeneus non-commutative polynomial p_{ij} of degree $i + j$ such that $x^i T^j_\sigma = p_{ij}(B, B')$. Then, by (9)–(11),

$$|\langle \phi_k, x^i T^j_\sigma \phi \rangle_\sigma| \le C_{ij} (1+k)^{m/2} \sum_{|\ell - k| \le m} |c_\ell| \tag{23}$$

for some $C_{ij} > 0$ independent of ϕ.

Assume that $\sigma \ge 0$. By (22), (23) and Theorem 2.1-(i), there is some $C'_{ij} > 0$, independent of ϕ and x_0, so that

$$\begin{aligned} |x_0|^\sigma\,|(x^i T^j_\sigma \phi)(x_0)| &\le |x_0|^\sigma \sum_k |\langle \phi_k, x^i T^j_\sigma \phi \rangle_\sigma|\,|\phi_k(x_0)| \\ &= \sum_k |\langle \phi_k, x^i T^j_\sigma \phi \rangle_\sigma|\,|\xi_k(x_0)| \le C'_{ij}\,\|\phi\|_{W^{m'}_\sigma} \end{aligned} \tag{24}$$

for all $x_0 \in \mathbb{R}$. Hence $\|\phi\|_{\mathcal{S}^m_\sigma} \le C' \|\phi\|_{W^{m'}_\sigma}$ for some $C' > 0$ independent of ϕ.

Now suppose that $\sigma < 0$. From (22), (23), and Theorem 2.1-(ii),(iii), it follows that there is some $C'_{ij} > 0$, independent of ϕ and x_0, so that

$$|(x^i T^j_\sigma \phi)(x_0)| \le \sum_k |\langle \phi_k, x^i T^j_\sigma \phi \rangle_\sigma|\,|\phi_k(x_0)| \le C'_{ij}\,\|\phi\|_{W^{m'}_\sigma}$$

if $\phi \in \mathcal{S}_{\sigma,\mathrm{ev/odd}}$, $i + j$ is even/odd and $|x_0| \le 1$; and $|x_0|^\sigma\,|(x^i T^j_\sigma \phi)(x_0)| \le C'_{ij}\,\|\phi\|_{W^{m'}_\sigma}$ otherwise, like in (24). Therefore $\|\phi\|_{\mathcal{S}^m_\sigma} \le C' \|\phi\|_{W^{m'}_\sigma}$ for some some $C' > 0$ independent of ϕ. □

As suggested by (15), consider the mapping $c = (c_k) \mapsto \Xi(c) = (d_\ell)$, where c is odd and $\Xi(c)$ is even with

$$d_\ell = \sum_{k\in\{\ell+1,\ell+3,\dots\}} (-1)^{\frac{k-\ell-1}{2}} \sqrt{\frac{(k-1)(k-3)\cdots(\ell+2)2s}{(k+2\sigma)(k-2+2\sigma)\cdots(\ell+1+2\sigma)}}\, c_k$$

for ℓ even, assuming that this series is convergent.

Lemma 4.12. Ξ *defines a bounded map* $\ell^2_{m',\mathrm{odd}} \to \mathcal{C}_{m,\mathrm{ev}}$ *if* $m' - m > 1$.

Proof. By the Cauchy-Schwartz inequality,

$$\begin{aligned}
\|d\|_{\mathcal{C}_m} &= \sup_\ell \sum_{k\in\{\ell+1,\ell+3,\dots\}} \sqrt{\frac{(k-1)(k-3)\cdots(\ell+2)2s}{(k+2\sigma)(k-2+2\sigma)\cdots(\ell+1+2\sigma)}}\, |c_k|\,(1+\ell)^m \\
&\le \sqrt{2s} \sup_\ell \sum_{k\in\{\ell+1,\ell+3,\dots\}} |c_k|\,(1+\ell)^m \\
&\le \sqrt{2s}\, \|c\|_{\ell^2_{m'}} \sup_\ell \left(\sum_{k\in\{\ell+1,\ell+3,\dots\}} (1+k)^{-m'} (1+\ell)^m \right)^{1/2} \\
&\le \sqrt{2s}\, \|c\|_{\ell^2_{m'}} \left(\sum_k (1+k)^{m-m'} \right)^{1/2},
\end{aligned}$$

where the last series is convergent since $m - m' < -1$. □

Corollary 4.13. x^{-1} *defines a bounded operator* $\mathcal{S}^{m'}_{\sigma,\mathrm{odd}} \to \mathcal{S}^m_{\sigma,\mathrm{ev}}$ *if* $2m' > m + 7$.

Proof. Since $2m' > m + 7$, there are $m_1, m_3 \in \mathbb{N}$ and $m_2 \in \mathbb{R}_+$ such that

$$m' - m_3 \ge 1\,, \quad m_3 - m_2 > 1\,, \quad 2m_2 - m_1 > 1\,, \quad m_1 - m > 1\,.$$

Then, by Propositions 4.3 and 4.4, Lemmas 4.7 and 4.12, and using the "Fourier coefficients" mapping, we get the following composition of bounded maps:

$$\mathcal{S}^{m'}_{\sigma,\mathrm{odd}} \hookrightarrow W^{m_3}_{\sigma,\mathrm{odd}} \to \ell^2_{m_3,\mathrm{odd}} \xrightarrow{\Xi} \mathcal{C}_{m_2,\mathrm{ev}} \hookrightarrow \ell^2_{m_1,\mathrm{ev}} \to W^{m_1}_{\sigma,\mathrm{ev}} \hookrightarrow \mathcal{S}^m_{\sigma,\mathrm{ev}}\,.$$

By (15), this composite is an extension of the map $x^{-1} : \mathcal{S}_{\mathrm{odd}} \to \mathcal{S}_{\mathrm{ev}}$. □

Question 4.14. Is it possible to prove Corollary 4.13 without using (15)?

Corollary 4.15. x^{-1} *defines a continuous operator* $\mathcal{S}_{\sigma,\mathrm{odd}} \to \mathcal{S}_{\sigma,\mathrm{ev}}$.

Lemma 4.16. $\mathcal{S}^{M_{m,\mathrm{ev/odd}}}_{\sigma,\mathrm{ev/odd}} \subset \mathcal{S}^m_{w,\sigma,\mathrm{ev/odd}}$ *continuously for all* m*, where*

$$M_{0,\mathrm{ev/odd}} = 0\,, \quad M_{1,\mathrm{ev}} = 1\,, \quad M_{1,\mathrm{odd}} = 4\,, \quad M_{2,\mathrm{ev/odd}} = 5\,,$$

$$M_{m,\mathrm{ev/odd}} = \begin{cases} m+4 & \text{if } m \text{ is even/odd} \\ m+3 & \text{if } m \text{ is odd/even} \end{cases} \qquad (m \ge 3)\,.$$

Proof. We proceed by induction on m. The case $m = 0$ was already indicated in the proof of Lemma 3.8. Now, take any $m > 0$ and assume that the result holds for $m - 1$.

For $\phi \in C^\infty_{\mathrm{ev}}$, $i + j \le m$ with $j > 0$ and $x \in \mathbb{R}$, we have $|x^i \phi^{(j)}(x)| = |x^i (T_\sigma\phi)^{(j-1)}(x)|$ with $T_\sigma\phi \in C^\infty_{\mathrm{odd}}$, obtaining $\|\phi\|_{\mathcal{S}^m_{w,\sigma}} \le \|T_\sigma\phi\|_{\mathcal{S}^{m-1}_{w,\sigma}}$. But, by the induction hypothesis and because $M_{m,\mathrm{ev}} = M_{m-1,\mathrm{odd}} + 1$, there are some $C, C' > 0$, independent of ϕ, such that $\|T_\sigma\phi\|_{\mathcal{S}^{m-1}_{w,\sigma}} \le C\, \|T_\sigma\phi\|_{\mathcal{S}^{M_{m-1,\mathrm{odd}}}_\sigma} \le C'\, \|\phi\|_{\mathcal{S}^{M_{m,\mathrm{ev}}}_\sigma}$.

For $\phi \in C^\infty_{\mathrm{odd}}$, let $\psi = x^{-1}\phi$, and take i, j and x as above. We have

$$\left|x^i\phi^{(j)}(x)\right| \le \left|x^i(T_\sigma\phi)^{(j-1)}(x)\right| + 2\,|\sigma|\,\left|x^i\psi^{(j-1)}(x)\right|$$

with $T_\sigma\phi,\psi \in C^\infty_{\mathrm{ev}}$, obtaining $\|\phi\|_{\mathcal{S}^m_{w,\sigma}} \le \|T_\sigma\phi\|_{\mathcal{S}^{m-1}_{w,\sigma}} + 2\,|\sigma|\,\|\psi\|_{\mathcal{S}^{m-1}_{w,\sigma}}$. But, by the induction hypothesis, Corollary 4.13, and since $M_{m,\mathrm{odd}} \ge M_{m-1,\mathrm{ev}} + 1$ and $2M_{m,\mathrm{odd}} > M_{m-1,\mathrm{ev}} + 7$, there are some $C, C' > 0$, independent of ϕ, such that

$$\begin{aligned}\|T_\sigma\phi\|_{\mathcal{S}^{m-1}_{w,\sigma}} + 2\,|\sigma|\,\|\psi\|_{\mathcal{S}^{m-1}_{w,\sigma}} &\le C\left(\|\phi'\|_{\mathcal{S}_\sigma^{M_{m-1,\mathrm{ev}}}} + \|\psi\|_{\mathcal{S}_\sigma^{M_{m-1,\mathrm{ev}}}}\right)\\ &\le C'\,\|\phi\|_{\mathcal{S}_\sigma^{M_{m,\mathrm{odd}}}}\;. \quad\square\end{aligned}$$

Corollary 4.17. *$\mathcal{S}^{M_{m,\mathrm{ev/odd}}}_{\sigma,\mathrm{ev/odd}} \subset \mathcal{S}^m_{\mathrm{ev/odd}}$ continuously for all m, where*

$$M_{0,\mathrm{ev}} = 1\;,\quad M_{0,\mathrm{odd}} = 4\;,\quad M_{1,\mathrm{ev/odd}} = 5 \qquad (\sigma \le 0)\;,$$
$$M_{0,\mathrm{ev/odd}} = 5 \qquad (0 < \sigma \le 1)\;,$$
$$M_{m,\mathrm{ev/odd}} = \begin{cases} m' + 5 & \text{if } m' \text{ is odd/even}\\ m' + 4 & \text{if } m' \text{ is even/odd}\end{cases} \qquad \left(m' = m + \frac{1}{2}\lceil\sigma\rceil(\lceil\sigma\rceil + 1) \ge 2\right)\;.$$

Proof. This follows from Lemmas 3.2, 3.4 and 4.16. $\square$

Corollary 4.18. *$\mathcal{S}_\sigma = \mathcal{S}$ as Fréchet spaces.*

Proof. This is a consequence of Corollaries 3.10 and 4.17 $\square$

Theorems 1.1 and 1.2 follow from Corollaries 3.10 and 4.17 and Propositions 4.3 and 4.4.

5. Perturbations of H on $\mathbb{R}_+$

We consider perturbations of H on $\mathbb{R}_+$ obtained by restricting the even component of L to $\mathbb{R}_+$, and then conjugating it by the operator of multiplication by a positive function defined almost everywhere on $\mathbb{R}_+$.

Since the function $|x|^{2\sigma}$ is even, the decomposition $\mathcal{S} = \mathcal{S}_{\mathrm{ev}} \oplus \mathcal{S}_{\mathrm{odd}}$ extends to an orthogonal decomposition

$$L^2(\mathbb{R}, |x|^{2\sigma}\,dx) = L^2_{\mathrm{ev}}(\mathbb{R}, |x|^{2\sigma}\,dx) \oplus L^2_{\mathrm{odd}}(\mathbb{R}, |x|^{2\sigma}\,dx)\;.$$

Let $L_{\mathrm{ev/odd}}$ and $\mathcal{L}_{\mathrm{ev/odd}}$, or $L_{\sigma,\mathrm{ev/odd}}$ and $\mathcal{L}_{\sigma,\mathrm{ev/odd}}$, denote the components of L and $\mathcal{L}$ with respect to these decompositions. $L_{\mathrm{ev/odd}}$ is essentially self-adjoint in $L^2_{\mathrm{ev/odd}}(\mathbb{R}, |x|^{2\sigma}\,dx)$, and its self-adjoint extension is $\mathcal{L}_{\mathrm{ev/odd}}$, which satisfies an obvious version of Corollary 1.3.

Fix an open subset $U \subset \mathbb{R}_+$ of full Lebesgue measure. Let $\mathcal{S}_{\mathrm{ev/odd},U} \subset C^\infty(U)$ denote the linear subspace of restrictions to U of the functions in $\mathcal{S}_{\mathrm{ev/odd}}$. The restriction to U defines a linear isomorphism $\mathcal{S}_{\mathrm{ev/odd}} \cong \mathcal{S}_{\mathrm{ev/odd},U}$, and a unitary isomorphism $L^2_{\mathrm{ev/odd}}(\mathbb{R}, |x|^{2\sigma}\,dx) \cong L^2(\mathbb{R}_+, 2x^{2\sigma}\,dx)$. Via these isomorphisms, $L_{\mathrm{ev/odd}}$ corresponds to an operator $L_{\mathrm{ev/odd},U}$ on $\mathcal{S}_{\mathrm{ev/odd},U}$, and $\mathcal{L}_{\mathrm{ev/odd}}$ corresponds to a self-adjoint operator $\mathcal{L}_{\mathrm{ev/odd},+}$ in $L^2(\mathbb{R}_+, x^{2\sigma}\,dx)$; the more explicit notation $L_{\sigma,\mathrm{ev/odd},U}$ and $\mathcal{L}_{\sigma,\mathrm{ev/odd},+}$ may be used. Let $\phi_{k,U} = \phi_k|_U$, whose norm in $L^2(\mathbb{R}_+, x^{2\sigma}\,dx)$ is $1/\sqrt{2}$.

Going one step further, for any positive function $h \in C^2(U)$, the multiplication by h defines a unitary isomorphism $h : L^2(\mathbb{R}_+, x^{2\sigma}\,dx) \to L^2(\mathbb{R}_+, x^{2\sigma}h^{-2}\,dx)$. Thus $hL_{\mathrm{ev},U}h^{-1}$, with domain $h\,\mathcal{S}_{\mathrm{ev},U}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, x^{2\sigma}h^{-2}\,dx)$,

and its self-adjoint extension is $h\mathcal{L}_{\mathrm{ev},+}h^{-1}$. Via these unitary isomorphisms, we get an obvious version of Corollary 1.3 for $h\mathcal{L}_{\mathrm{ev},+}h^{-1}$. By using

$$\left[\frac{d}{dx},h\right]=h'\ ,\quad \left[\frac{d^2}{dx^2},h\right]=2h'\frac{d}{dx}+h''\ , \tag{25}$$

it easily follows that $hL_{\mathrm{ev},U}h^{-1}$has the form of P in Theorem 1.4 with $f_1\in C^1(U)$ and $f_2\in C(U)$. Then Theorem 1.4 is a consequence of the following.

Lemma 5.1. *For $\sigma>-1/2$, a positive function $h\in C^2(U)$, and $P=H-2f_1\frac{d}{dx}+f_2$ with $f_1\in C^1(U)$ and $f_2\in C(U)$, we have $P=hL_{\sigma,\mathrm{ev},U}h^{-1}$ on $h\,\mathcal{S}_{\mathrm{ev},U}$ if and only if f_1, f_2 and h satisfy the conditions of Theorem 1.4.*

Proof. By (25),

$$h^{-1}Ph=H-2(h^{-1}h'+f_1)\frac{d}{dx}-h^{-1}h''-2h^{-1}f_1h'+f_2\ .$$

So $P=hL_{\sigma,\mathrm{ev},U}h^{-1}$ if and only if $h^{-1}h'=\sigma x^{-1}-f_1$ and $f_2=h^{-1}h''+2h^{-1}h'f_1$, which are easily seen to be equivalent to the conditions of Theorem 1.4. □

Remark 3. By (25), we get an operator of the same type if h and $\frac{d}{dx}$ is interchanged in the operator P of Theorem 1.4.

Remark 4. By using (25) with $h=x^{-1}$ on $\mathbb{R}_+$, it is easy to check that $L_{\sigma,\mathrm{odd},\mathbb{R}_+}=xL_{1+\sigma,\mathrm{ev},\mathbb{R}_+}x^{-1}$ on $\mathcal{S}_{\mathrm{odd},\mathbb{R}_+}=x\,\mathcal{S}_{\mathrm{ev},\mathbb{R}_+}$. So no new operators are obtained with this process by using $L_{\sigma,\mathrm{odd}}$ instead of $L_{\sigma,\mathrm{ev}}$.

References

[1] J.A. Álvarez López and M. Calaza, *Application of the method of Bonan-Clark to the generalized Hermite polynomials*, arXiv:1101.5022v8 [math.SP], 2012.
[2] ———, *Witten's perturbation on strata*, arXiv:1205.0348v1 [math.DG], 2012.
[3] R. Askey and S. Wainger, *Mean convergence of expansions in Laguerre and Hermite series*, Amer. J. Math. **87** (1965), 695–708.
[4] S.S. Bonan and D.S. Clark, *Estimates of the Hermite and the Freud polynomials*, J. of Approx. Theory **63** (1990), 210–224.
[5] J.P. Brasselet, G. Hector, and M. Saralegi, *L^2-cohomologie des espaces stratifiés*, Manuscripta Math. **76** (1992), 21–32.
[6] J. Brüning and M. Lesch, *Hilbert complexes*, J. Funct. Anal. **108** (1992), 88–132.
[7] J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Geometry of the Laplace Operator (Univ. Hawaii, Honolulu, Hawaii, 1979) (Providence, R.I., 1980), Proc. Sympos. Pure Math., vol. XXXVI, Amer. Math. Soc., 1980, pp. 91–146.
[8] ———, *Spectral geometry of singular Riemannian spaces*, J. Differential Geom. **18** (1983), 575–657. MR 85d:58083
[9] T.S. Chihara, *Generalized Hermite polynomials*, Ph.D. thesis, Purdue University, 1955.
[10] ———, *An introduction to orthogonal polynomials*, Mathematics and its Applications, vol. 13, Gordon and Breach Science Publishers, New York-London-Paris, 1978.
[11] D. Dickinson and S.A. Warsi, *On a generalized Hermite polynomial and a problem of Carlitz*, Boll. Un. Mat. Ital. **18** (1963), 256–259.
[12] C.F. Dunkl, *Reflection groups and orthogonal polynomials on the sphere*, Math. Z. **197** (1988), 33–60.
[13] ———, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311** (1989), 167–183.
[14] ———, *Operators commuting with coxeter group actions on polynomials*, Invariant Theory and Tableaux (Minneapolis, MN, 1988) (New York) (D. Stanton, ed.), IMA Vol. Math. Appl., vol. 19, Springer, 1990, pp. 107–117.
[15] ———, *Integral kernels with reflection group invariants*, Canad. J. Math. **43** (1991), 1213–1227.

[16] ———, *Hankel transforms associated to finite reflection groups*, Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991) (Providence, RI), Contemp. Math., vol. 138, Amer. Math. Soc., 1992, pp. 123–138.
[17] ———, *Symmetric functions and B_N-invariant spherical harmonics*, J. Phys. A **35** (2002), 10391–10408.
[18] M. Dutta, S.K. Chatterjea, and K.L. More, *On a class of generalized Hermite polynomials*, Bull. Inst. Math. Acad. Sinica **3** (1975), 377–381.
[19] A. Erdélyi, *Asymptotic forms for Laguerre polynomials*, J. Indian Math. Soc. **24** (1960), 235–250.
[20] M. Goresky and R. MacPherson, *Morse theory and intersection homology theory*, Analysis and Topology on Singular Spaces, II, III, Luminy, 1981, Astérisque, vol. 135-192, 1983, pp. 135–192.
[21] L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis*, second ed., Grundlehren der mathematischen Wissenschaften, no. 256, Springer-Verlag, Berlin, 1990.
[22] J.N. Mather, *Notes on topological stability*, Mimeographed Notes, Hardvard University, 1970.
[23] B. Muckenhoupt, *Asymptotic forms for Laguerre polynomials*, Proc. Amer. Math. Soc. **24** (1970), 288–292.
[24] ———, *Mean curvergence of Laguerre and Hermite series. II*, Trans. Amer. Math. Soc. **147** (1970), 433–460.
[25] N. Nagase, *L^2-cohomology and intersection cohomology of stratified spaces*, Duke Math. J. **50** (1983), 329–368.
[26] ———, *Sheaf theoretic L^2-cohomology*, Adv. Stud. Pure Math. **8** (1986), 273–279.
[27] A. Nowak and K. Stempak, *Imaginary powers of the Dunkl harmonic oscillator*, SIGMA Symmetry Integrability Geom. Methods Appl. **5** (2009), Paper 016, 12 pp.
[28] ———, *Riesz transforms for the Dunkl harmonic oscillator*, Math. Z. **262** (2009), 539–556.
[29] J. Roe, *Elliptic operators, topology and asymptotic methods*, second ed., Pitman Research Notes in Mathematics, vol. 395, Addison Wesley Longman Limited, Edinburgh Gate, Harlow, Essex CM20 2JE, England, 1998.
[30] M. Rosenblum, *Generalized Hermite polynomials and the Bose-like oscillator calculus*, Nonselfadjoint operators and related topics (Beer Sheva, 1992) (Basel) (A. Feintuch and I. Gohberg, eds.), Oper. Theory Adv. Appl., vol. 73, Birkhäuser, 1994, pp. 369–396.
[31] M. Rösler, *Generalized Hermite polynomials and the heat equation for Dunkl operators*, Comm. Math. Phys. **192** (1998), 519–542.
[32] ———, *Dunkl operators: theory and applications*, Orthogonal polynomials and special functions (Leuven, 2002) (Berlin) (E. Koelink and W. Van Assche, eds.), Lecture Notes in Math., vol. 1817, Springer, 2003, pp. 93–135.
[33] G. Szegö, *Orthogonal polynomials*, fourth ed., Colloquium Publications, vol. 23, Amer. Math. Soc., Providence, RI, 1975.
[34] R. Thom, *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc. **75** (1969), 240—284.
[35] A. Verona, *Stratified mappings—structure and triangulability*, Lecture Notes in Math., vol. 1102, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.
[36] S. Watanabe, *An embedding theorem of Sobolev type for an operator with singularity*, Proc. Amer. Math. Soc. **125** (1997), 839–848.
[37] ———, *A generalized Fourier transform and its applications*, Integral Transforms Spec. Funct. **13** (2002), 321–344.
[38] ———, *An embedding theorem of Sobolev type*, Integral Transforms Spec. Funct. **15** (2004), 369–374.
[39] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. **17** (1982), 661–692.

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